Research Report

In this section research work with experimental results or theoretical values is published in brief (not full paper) more as preliminary result.

Guidelines for research report preparation

In addition to what has been stated in Manuscript Preparation, the following points are to be noted:

Abstract with no mathematical expression and reference could be provided.

Diagrams, mathematical expressions, tables and references shall be minimum. Broad discipline (eg. Physics/Astrophysics/Chemistry/Mathematics/Biochemistry) with the subdivision under which a research report comes should be indicated.

Well-finished diagrams should be sent and figures bearing numbers with suitable captions should be sent separately. Rough sketches of figures and diagrams are not acceptable.

Hand-written mathematical expressions must be written by black ink unambiguously leaving enough space at the top and bottom of each line.

References shall have to be quoted as practised in this medium.

Key words if not mentioned by the author(s) will not be provided. Running title, if not provided, will be given.

Research Report — not written as per above instructions — will be returned for rewriting, even though it is approved by the referee(s).

Mathematics

A Note Concerning Congruence Relations on Austere Commutative Semirings

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A *semiring* is a nonempty set R on which operations of addition and multiplication are defined satisfying the following conditions :

1. $(R, +)$ is a commutative monoid with identity element 0 ;
2. $(R, .)$ is a monoid with identity element 1 ;
3. Multiplication distributes over addition from either side ;
4. $0r = 0 = r0$ for all $r \in R$;
5. $1 \neq 0$.

Note that a ring is just a semiring in which each element has an additive inverse. At the other extreme, a semiring is *zerosumfree* if and only if no nonzero element has an additive inverse. A semiring is *commutative* if multiplication is commutative. Notation and terminology concerning semirings will be followed as given by Golan[1].

A nonempty proper subset I of a semiring R is an *ideal* of R if and only if $a, b \in I$ and $r \in R$ imply $a + b, ra, ar \in I$. An ideal I or R is *subtractive* if and only if it has the additional property that $a \in I$ and $a + b \in I$ imply that $b \in I$. Not every ideal of a semiring need be subtractive. However, subtractive ideals are particularly important since they are precisely the kernels of morphisms between semirings. This is the reason that such ideals are sometimes called "kernel ideals" or, simply, "k-ideals". If R is a ring then, of course, all ideals are subtractive. The family of all subtractive ideals of a semiring R is closed under taking arbitrary intersections. Therefore, given an ideal I of R, we can consider the intersection of all subtractive ideals containing I (using the standard convention that the intersection of an empty set of ideals of R is R itself) to obtain a subtractive ideal called the *subtractive closure* of I in R. Following Golan[1], we denote the subtractive closure of I by $0/I$. To understand this notation, we note that every ideal I of a semiring R defines a congruence relation $\equiv_I$ on R, called the *Bourne relation*, given by $r \equiv_I S$ if and only if there exist elements a and b of I satisfying $r + a = s + b$. The subtractive ideal $0/I$ is just the congruence class of 0 relative to this relation.

A semiring R is *austere* if $\{0\}$ is its sole subtractive ideal. Several characterizations of such semirings exist. For example, a commutative semiring R is austere if and only if for each $0 \neq r \in R$ there exist elements a and b of R satisfying $ar + 1 = br$. See Proposition 5.41 of Golan[1]. Every austere commutative semiring is *entire*, namely it has no zero divisors. Indeed, if r, s are elements of such a semiring satisfying $rs = 0$ with $r \neq 0$ then there exist elements a and b of R satisfying $ar + 1 = br$ and so $s = (ar + 1)s = (br)s = 0$. See Golan[1] Proposition 5.18.

1. **Proposition.** *An austere commutative semiring R is either zerosumfree or a field.*

Proof. Let R be an austere commutative semiring which is not zerosumfree and let I be the set of all elements of R having an additive inverse. By assumption, $I \neq \{0\}$ and it is straighforward to see that I is a subtractive ideal of R. By austerity, we must then have $I = R$ and so R is in fact a ring. If $0 \neq a \in R$ then $\{ra \mid r \in R\}$ is a subtractive ideal of R not equal to $\{0\}$ and so, again by austerity, it must equal R. Therefore a must have a multiplicative inverse as well, proving that R is in fact a field.

Let R be an austere commutative semiring and define a relation $\sim$ on R defined by the condition that $a \sim b$ if and only if $a = b = 0$ or $ab \neq 0$. This is an equivalence relation on R. Indeed, clearly $0 \sim 0$. If $0 \neq a \in R$ then $a^2 \neq 0$ by entirety so $a \sim a$. It is also immediately obvious that $a \sim b$ implies $b \sim a$. Now assume $a \sim b$ and $b \sim c$. If one of a, b, c equals 0 so do the others, and so we have $a \sim c$. Assume therefore that all three elements are nonzero. Then $ab \neq 0 \neq bc$. Again, by entirety, we must have $ab^2c \neq 0$ and so, in particular, $ac \neq 0$, ie $a \sim c$. Thus $\sim$ is an equivalence relation on R.

2. **Proposition**. *If R is an austere commutative semiring then $\sim$ is a congruence relation whenever* R *is zerozsumfree. The converse holds if R has more than two elements.*

Proof. Assume that R is *zerosumfree* and suppose that $a \sim b$ and $c \sim d$ in R. If one of these four elements equals 0 then surely $ac \sim bd$ and $a + c \sim b + d$ so, without loss of generality, we can assume that all of them are nonzero. Then $ab \neq 0 \neq cd$ and so, by entirety, we conclude that $abcd \neq 0$ ie, $ac \sim bd$. Since R is *zerosumfree*, we have $a + c \neq 0 \neq b + d$ and so, by entirety, $(a + c)(b + d) \neq 0$. Therefore $a + c \sim b + d$.

Conversely, assume that R has more than two elements and that $\sim$ is a congruence relation. If R is not *zerosumfree* then, by Proposition 1, it is a field. In this case, $\sim$ must either be the improper congruence ($a \sim b$ for all $a, b \in R$) or the trivial congruence ($a \sim b$ if and only if $a = b$). The former cannot be the case since $0 \sim 1$, while the latter cannot be the case since $a \sim 1$ for any $a \in R \setminus \{0, 1\}$. Thus R must be *zerosumfree*, as desired.

The set $Cong(R)$ of all congruence relations on a semiring R forms a complete lattice. In particular, it has a partial order given as follows : $\equiv_1 \leq \equiv_2$ if and only if $a \equiv_1 b$ implies $a \equiv_2 b$ for all $a, b \in R$. If R is a field then, as we have already noticed, the set $Cong(R)$ has precisely two elements and so, trivially, has a unique maximal proper element. The same is in fact true for austere commutative semirings.

3. **Proposition.** *If R is an austere commutative semiring then $Cong(R)$ has a unique maximal proper element.*

Proof. The result is immediate if R is a field or if R has but two elements, so assume that such is not the case. Then, by Proposition 2, $\sim$ is a congruence relation on R. Let $\equiv$ be an arbitrary nontrivial congruence relation on R. Then $I = \{ a \in R \mid a \equiv 0 \}$ is a subtractive ideal of R and so must equal $\{0\}$. Thus $a \equiv 0$ implies $a = 0$ and so $a \sim 0$, If $a \equiv b$ where $a \neq 0 \neq b$, then $ab \neq 0$ by entirety and so $a \sim b$. Thus $\equiv \leq \sim$ in $Cong(R)$.

Any proper congruence $\equiv$ on a semiring R defines a factor semiring $R/\equiv$ and a canonical surjective morphism of semirings $\gamma : r \rightarrow r/\equiv$ which takes each element of R to its corresponding congruence class. The kernel of γ is the class $0/\equiv$. This is a subtractive ideal of R and so, in particular, we note if R is an austere semiring then every morphism of semirings $R \rightarrow S$ has kernel equal to $\{0\}$. On the other hand, Proposition 2 shows that if R is an austere commutative, semiring which is not a field and which has more than two elements then there exists a nontrivial proper congruence on R, namely $\sim$. Thus we arrive at the following conclusion.

4. **Proposition.** *If R is an austere commutative semiring which is not a field and which has more than two elements then there exists a morphism of semirings $R \rightarrow S$ having kernel $\{0\}$ which is not injective.*

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REFERENCE

1. J S Golan, The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science, Longman Scientific & Technical, Harlow, 1992.